

Multi-agent Robust Consensus Convergence Analysis and Application*

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Abstract

The paper investigates consensus problem for continuous-time multi-agent systems with time-varying communication graphs subject to process noises. Borrowing the ideas from input-to-state stability (ISS) and integral input-to-state stability (iISS), robust consensus and integral robust consensus are defined with respect to L_∞ and L_1 norms of the disturbance functions, respectively. Sufficient and/or necessary connectivity conditions are obtained for the system to reach robust consensus or integral robust consensus, which answer the question: how much communication capacity is required for a multi-agent network to converge despite certain amount of disturbance. The ϵ -convergence time is then obtained for the network as a special case of the robustness analysis. The results are based on quite general assumptions on switching graph, weights rule and noise regularity. In addition, as an illustration of the applicability of the results, distributed event-triggered coordination is studied.

Keywords: Multi-agent systems, Robust consensus, Joint connection, Convergence rate, Event-triggered coordination

1 Introduction

Coordination of multi-agent networks has attracted a significant amount of attention in the past few years, due to its broad applications in various fields of science including physics, engineering, biology, ecology and social science [6, 24, 20, 42, 19]. Distributed control using neighboring information flow has been shown to ensure collective tasks such as formation, flocking, rendezvous, and aggregation [14, 35, 7, 26].

Central to multi-agent coordination study is the study of consensus, or state agreement, which requires that all agents achieve the desired relative position and the same velocity. Consensus seeking is extensively studied in the literature for both continuous-time and discrete-time models [29, 24, 42, 21, 22, 15, 16, 9, 35, 23, 25]. Recently also asynchronous event-triggered sampling for such consensus-seeking multi-agent system was studied [44, 47, 48]. Researchers are not only concerned with what connectivity conditions can guarantee consensus, but also

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with the convergence rate: how fast the network reaches a consensus under certain connectivity assumptions [35, 16, 30, 31, 28].

Consensus algorithms are usually carried out over an underlying communication network. Thus, connectivity of this communication graph plays a key role in consensus analysis. Various connectivity conditions have been used to describe frequently switching topologies in different cases. The “joint connection”, i.e., the union of graphs over a time interval, and similar concepts are important in the analysis of consensus stability with time-dependent topology. Uniformly joint connectedness, which requests the joint connection is connected for all intervals which are longer than some positive constant, has been employed for different consensus problems from discrete-time to continuous-time agent dynamics, from directed to undirected interconnection topologies [29, 24, 35, 17, 18]. [29] studied the distributed asynchronous iterations, while [24] proved the consensus of a simplified Vicsek model. Furthermore, [17] and [18] investigated the jointly-connected coordination for second-order agent dynamics. A nonlinear continuous-time model was discussed in [35] with directed communications, in which convergence to a consensus is shown to be uniform within bounded initial conditions. On the other hand, $[t, \infty)$ -joint connection requires the joint connection is connected for infinitely many disjoint intervals in $[0, \infty]$, which was discussed in [42] for consensus seeking of discrete-time agents. This connectivity concept was then extended in continuous-time distributed control analysis for target set convergence and state agreement in [26].

Communication over networks is typically unreliable and with channel noise, which has attracted researchers to look at the robustness of consensus algorithms [10, 11, 12, 13, 39]. In [41], robustness performance was discussed for average consensus algorithms. Then in [10, 11], robust consensus was studied under directed communication graphs for discrete-time systems. For continuous-time multi-agent systems, robustness of consensus was established by an H_2 bound for networks of single integrators with a fixed directed communication graph [40]. In [39], robust consensus with diverse input delays and asymmetric interconnection perturbations was discussed for a second-order leader-following model. Recently, an optimal synchronization protocol was studied for discrete-time double integrators subject to process noise [38].

Clearly, robustness of consensus algorithms subject to noise highly relies on the convergence rate for the algorithm in the absence of noise. The concept of ϵ -convergence time, was introduced to quantify the convergence speed for discrete-time consensus algorithms. It is defined as the minimum time steps required for the network to reach a certain level of consensus captured by a parameter ϵ . Bounds of ϵ -convergence time have been widely established in the literature for first-order discrete-time dynamics [28, 16, 30, 31], and recently a sharp bound was presented in [31] indicating that the convergence time is of order $O(n^2B)$, where n is the number of nodes in the network and B is a lower bound for the time interval in the definition of uniformly joint connectedness. Few results have been obtained on the convergence rates for continuous-time multi-agent systems reaching a consensus with general joint connectivity assumptions. The robustness consensus analysis for continuous-time systems a challenging problem. A quantitative answer to how much noise can be dealt with by how much communication is still missing.

The primary aim of this paper is to establish the convergence towards a consensus for first-order, continuous-time multi-agent systems with communication noise for general directed and time-varying interconnection graphs. Borrowing ideas from input-to-state stability (ISS) and integral input-to-state stability (iISS) [32, 33], we define robust consensus and integral robust consensus for the relative-position-based continuous-time coordination protocol which was first

introduced in [20]. We present explicit convergence bounds for the system with respect to L_∞ and L_1 norms of the disturbances. Sufficient and necessary connectivity conditions are obtained for the system to reach robust consensus or integral robust consensus, respectively, for directed and bidirectional communications. Consequently, the upper bounds for the ϵ -convergence time are established under uniform or non-uniform joint connections, as a simple special case of the robust consensus analysis. To the best of our knowledge, the results from this analysis are the first to show that consensus is reached exponentially in t with uniformly joint connected graphs, while “exponentially” in the times that the joint graph are connected with $[t, \infty)$ -joint connection for the considered systems. Different with most existing work, we build the whole analysis on a generalized integral assumption for the weight functions, which covers a large amount of common functions. Additionally, a class of event-triggered coordination rules is studied as the application of the robust consensus results.

The paper is organized as follows. In Section 2, some preliminary concepts are introduced. We set up the system model, present our standing assumptions and main results in Section 3. Then convergence analysis is carried out for directed and bidirectional graphs in Sections 4 and 5, respectively. We turn to event-triggered consensus in Section 6, as an application of the results obtained in previous sections. Finally, concluding remarks are given in Section 7.

2 Preliminaries

Here we introduce some notations and theories on directed graphs and Dini derivatives.

2.1 Directed Graphs

A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set \mathcal{V} of nodes and an arc set \mathcal{E} [2]. An element $e = (i, j) \in \mathcal{E}$ is called an *arc* from node $i \in \mathcal{V}$ and to node $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ of nodes v_i and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, n$, the sequence is called a (directed) *path* with *length* n , and if $v_0 = v_n$ a (directed) *cycle*. A path with no repeated nodes is called a *simple path*. A digraph without cycles is said to be *acyclic*.

A path from i to j is denoted as $i \rightarrow j$, and the length of $i \rightarrow j$ is denoted as $|i \rightarrow j|$. If there exists a path from node i to node j , then node j is said to be reachable from node i . Each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of \mathcal{G} . A digraph \mathcal{G} is said to be *strongly connected* if it contains path $i \rightarrow j$ and $j \rightarrow i$ for every pair of nodes i and j ; *quasi-strongly connected* if \mathcal{G} has a center [5, 35].

In this paper, we define the (generalized) *distance* from i to j , $d(i, j)$, as the length of a longest simple path $i \rightarrow j$ if j is reachable from i , and the (generalized) *diameter* of \mathcal{G} as $\max\{d(i, j) | i, j \in \mathcal{V}, j \text{ is reachable from } i\}$.

A digraph \mathcal{G} is said to be bidirectional if for every two nodes i and j , i is a neighbor of j if and only if j is a neighbor of i . Then *path*, *distance*, *diameter* can be similarly defined for \mathcal{G} by ignoring the direction of the arcs. A bidirectional graph \mathcal{G} is said to be *connected* if there is a path between any two nodes.

2.2 Dini Derivatives

The upper Dini derivative of a function $h : (a, b) \rightarrow \mathbb{R}$ at t is defined as

$$D^+h(t) = \limsup_{s \rightarrow 0^+} \frac{h(t+s) - h(t)}{s}$$

The next result is useful for the calculation of Dini derivatives [3, 35].

Lemma 1 *Let $V_i(t, x) : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ ($i = 1, \dots, n$) be C^1 and $V(t, x) = \max_{i=1, \dots, n} V_i(t, x)$. If $\mathcal{I}(t) = \{i \in \{1, 2, \dots, n\} : V(t, x(t)) = V_i(t, x(t))\}$ is the set of indices where the maximum is reached at t , then $D^+V(t, x(t)) = \max_{i \in \mathcal{I}(t)} \dot{V}_i(t, x(t))$.*

Notations: For a vector $z = (z_1, \dots, z_N)^T$ in \mathbb{R}^N , $|z|$ denotes the maximum norm, i.e., $|z| \doteq \max_{i=1, \dots, N} |z_i|$. When $z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N$ is a measurable function defined on $[0, +\infty)$, $\|z\|_\infty$ denotes the essential supremum of $\{|z(t)|, t \in [0, +\infty)\}$. Moreover, a function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a \mathcal{K} -class function if it is continuous, strictly increasing, and $\gamma(0) = 0$; a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is a \mathcal{KL} -class function if $\beta(\cdot, t)$ is of class \mathcal{K} for each fixed $t \geq 0$ and $\beta(s, t) \rightarrow 0$ decreasingly as $t \rightarrow \infty$ for each fixed $s \geq 0$.

3 Problem Statement and Main Results

This paper considers a multi-agent system with agent set $\mathcal{V} = \{1, \dots, N\}$, $N \geq 2$, for which the dynamics of each agent is a first-order integrator:

$$\dot{x}_i = u_i, \quad i = 1, \dots, N \quad (1)$$

where $x_i \in \mathbb{R}$ represents the state of agent i , and u_i is its control input. Let $x = (x_1, \dots, x_N)^T$.

3.1 Network, Dynamics and Assumptions

The communication in the network is modeled as a time-varying graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ with $\sigma : [0, +\infty) \rightarrow \mathcal{Q}$ a piecewise constant function, and \mathcal{Q} a finite set indicating all possible graphs. Node j is said to be a *neighbor* of i at time t when there is an arc $(j, i) \in \mathcal{E}_{\sigma(t)}$. Let $\mathcal{N}_i(\sigma(t))$ represent the set of agent i 's neighbors at time t . An assumption is given to the variation of $\mathcal{G}_{\sigma(t)}$.

A1. (Dwell Time) There is a lower bound constant $\tau_D > 0$ between two consecutive switching time instants of $\sigma(t)$.

Denote the joint graph of $\mathcal{G}_{\sigma(t)}$ in time interval $[t_1, t_2)$ with $t_1 < t_2 \leq +\infty$ as $\mathcal{G}([t_1, t_2)) = (\mathcal{V}, \cup_{t \in [t_1, t_2)} \mathcal{E}_{\sigma(t)})$. We introduce the following definitions on connectivity.

Definition 1 *(i) $\mathcal{G}_{\sigma(t)}$ is said to be uniformly (jointly) strongly connected (USC) if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is strongly connected for any $t \geq 0$.*

(ii) $\mathcal{G}_{\sigma(t)}$ is said to be uniformly (jointly) quasi-strongly connected (UQSC) if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T])$ is quasi-strongly connected for any $t \geq 0$.

(iii) Assume that $\mathcal{G}_{\sigma(t)}$ is bidirectional for any $t \geq 0$. $\mathcal{G}_{\sigma(t)}$ is said to be infinitely jointly connected (IJC) if $\mathcal{G}([t, \infty))$ is connected for any $t \geq 0$.

Let a piecewise continuous function $a_{ij}(t) > 0$ be the weight of arc $(j, i), i, j \in \mathcal{V}$. We study the following agent-dynamics with noise, which was first introduced in [20].

$$\dot{x}_i(t) = u_i(t) = \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(t)(x_j(t) - x_i(t)) + w_i(t), \quad i = 1, \dots, N. \quad (2)$$

where $w_i(t)$ is a function which describes disturbances.

We have the following assumption on the weight functions $a_{ij}(t)$.

A2. (Weights Rule) There are two constants $0 < a_* \leq a^*$ such that

$$a_* \leq \int_t^{t+\tau_D} a_{ij}(s) ds \leq a^*, \quad t \in \mathbb{R}^+.$$

Remark 1 Note that assumption A2 is much weaker than the compact assumption, which is widely used in the literature [24, 30, 31, 15, 16, 26, 27], requiring that the arc weights are restricted within a compact set, and therefore they always have positive lower and upper bound. Indeed, A2 allows us to deal with general weight functions like $a_{ij}(t) = |\sin t|$ which cannot be covered by the compact assumption.

Introduce $\mathcal{F} \doteq \{z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N : \|z\|_\infty < \infty, \text{ and } z \text{ is } C^0 \text{ except for a set with measure 0}\}$. In order to ensure the existence of the solutions of (2), we impose the following assumption on the regularity of the disturbance function $w(t) = (w_1(t), \dots, w_n(t))^T$.

A3. (Disturbance Regularity) $w(t) \in \mathcal{F}$.

We assume that assumptions A1-A3 are standing assumptions. Under Assumptions A1 and A2, the set of discontinuity points for the right hand side of equation (2) has measure zero. Therefore, the Caratheodory solutions of (2) exist for arbitrary initial conditions, and they are absolutely continuous functions that satisfy (2) for almost all t on the maximum interval of existence [1, 4]. In the following, each solution of (2) is considered in the sense of Caratheodory without explicit mention.

3.2 The Robust Consensus Problem

Consider (2) with initial condition $x(t_0) = (x_1(t_0), \dots, x_N(t_0))^T = x^0 \in \mathbb{R}^N, t_0 \geq 0$. Let

$$\bar{h}(t) \doteq \max_{i \in \mathcal{V}} \{x_i(t)\}, \quad \ell(t) \doteq \min_{i \in \mathcal{V}} \{x_i(t)\}$$

be the maximum and minimum state value at time t , respectively. Denote $\mathcal{H}(x(t)) = \bar{h}(t) - \ell(t)$. Inspired by the concepts of input-to-state stability (ISS) and integral input-to-state stability (iISS) [33, 32], we introduce the following definition.

Definition 2 (i) (L^∞ to L^∞) System (2) achieves a global robust consensus (GRC) if there exist a \mathcal{KL} -function β and a \mathcal{K} -function γ such that for all $w \in \mathcal{F}$ and initial conditions $x(t_0) = x^0$,

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x^0), t) + \gamma(\|w\|_\infty), \quad t \geq 0. \quad (3)$$

(ii) (L^2 to L^∞) System (2) achieves a global integral robust consensus (GIRC) if there exist a \mathcal{KL} -function β and a \mathcal{K} -function γ such that for all $w \in \mathcal{F}$ and initial conditions $x(t_0) = x^0$,

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x^0), t) + \int_0^t \gamma(|w(s)|) ds, \quad t \geq 0. \quad (4)$$

We also introduce the following definition on consensus.

Definition 3 (i) A global consensus (GC) is achieved for system (2) if for any initial condition $x(t_0) = x^0 \in \mathbb{R}^N$,

$$\lim_{t \rightarrow \infty} \mathcal{H}(x(t)) = 0$$

(ii) Assume that $\mathcal{F}_0 \subseteq \mathcal{F}$. Then a global asymptotic consensus (GAC) with respect to \mathcal{F}_0 is achieved for system (2) if $\forall x^0 \in \mathbb{R}^N, \forall w \in \mathcal{F}_0, \forall \varepsilon > 0, \forall c > 0, \exists T > 0$ such that $\forall t_0 \geq 0$,

$$\mathcal{H}(x^0) \leq c \quad \Rightarrow \quad \mathcal{H}(x(t)) \leq \varepsilon, \forall t \geq t_0 + T.$$

3.3 Main Results

The target of the paper is to establish proper connectivity conditions of the underlying communication graph which can ensure robust consensus or integral robust consensus.

For networks with general directed communication graphs, we have the following conclusions.

Theorem 1 System (2) achieves a GRC iff $\mathcal{G}_{\sigma(t)}$ is UQSC.

Theorem 2 System (2) achieves a GIRC if $\mathcal{G}_{\sigma(t)}$ is UQSC.

It has been shown in [32] that ISS implies iISS. Now we see from Theorems 1 and 2 that

$$GRC \iff UQSC \implies GIRC.$$

This leads to the fact that GRC implies GIRC, which is consistent with the ISS and iISS properties.

Next, when the communication graph is restricted to be bidirectional all the time, we have the following result.

Theorem 3 Assume that $\mathcal{G}_{\sigma(t)}$ is bidirectional for any $t \geq 0$. System (2) achieves a GIRC iff $\mathcal{G}_{\sigma(t)}$ is IJC.

This robust consensus problem is generally challenging due to the coupled agent dynamics, especially under directed communication and time-varying arc weights. Moreover, a common Lyapunov function is often missing when we consider joint connectivity conditions. To obtain the desired GRC or GIRC inequalities, we have to derive the convergence rate for consensus explicitly against the impact of the disturbance.

We will present the convergence analysis of the main results in Sections 4 and 5, respectively, for directed and bidirectional graphs.

Remark 2 In [27], a set tacking problem is studied for multi-agent systems guided by multiple leaders. Set input-to-state stability (SISS) and set integral input-to-state stability (SiISS) are used to describe the set convergence property. We see that the results obtained in this paper are consistent with the SISS and SiISS analysis in [27].

However, the convergence results in [27] cannot be applied to the model discussed in this paper. Note that leaderless consensus is usually a much harder problem than the leader-follower case, especially under time-varying communication graphs. In leader-follower model, the leader(s) can always be treated as center node and therefore the network has a very special topology. The main difficulty here lies in that the center node may be different for different time intervals and that its dynamics is also influenced by other nodes. As will be shown in the following discussions, the symmetry in the structure of $\mathcal{H}(t)$, plays a key role in the convergence analysis.

Additionally, different from [27] and most other existing works, the standing assumption on the weights rule in current paper, A2, is a much weaker condition than the usually applied.

4 Convergence: Directed Graphs

In this section, we establish convergence analysis for directed graphs.

The necessity statement of Theorem 1 follows from a similar argument which was used in [35]. Assume that $\mathcal{G}_{\sigma(t)}$ is not UQSC. Then for any $T_* > 0$ there exists $t_* \geq 0$ such that $\mathcal{G}([t_*, t_* + T_*])$ is not quasi-strongly connected. Consequently, there exists two distinct nodes i and j such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, where $\mathcal{V}_1 = \{\text{nodes from which } i \text{ is reachable in } \mathcal{G}([t_*, t_* + T_*])\}$ and $\mathcal{V}_2 = \{\text{nodes from which } j \text{ is reachable in } \mathcal{G}([t_*, t_* + T_*])\}$. Let $w_i(t) \equiv 0$ for $i \in \mathcal{V}_1$ and $w_i(t) \equiv 1$ for $i \in \mathcal{V}_2$ when $t \in [t_*, t_* + T_*]$. Let initial condition t_0 be t_* with $x_i(t_*) = 0, \forall i \in \mathcal{V}$. It is not hard to see that $\mathcal{H}(x(t_* + T_*)) = T_*$. Hence, GRC cannot be achieved since T_* can be arbitrarily large.

4.1 Key Lemmas

We first establish the following lemma indicating that the Dini derivative of $h(t)$ is bounded above by $|w(t)|$, and the Dini derivative of $\ell(t)$ is bounded below by $-|w(t)|$.

Lemma 2 For all $t \geq t_0 \geq 0$, we have

$$D^+ h(t) \leq |w(t)|; \quad D^+ \ell(t) \geq -|w(t)|$$

Proof. We prove $D^+h(t) \leq |w(t)|$. The other part can be proved similarly.

Let $\mathcal{I}(t)$ represent the set containing all the agents that reach the maximum in the definition of $h(t)$ at time t , i.e., $\mathcal{I}(t) = \{i \in \mathcal{V} \mid x_i(t) = h(t)\}$. Then according to Lemma 1, we obtain

$$D^+h(t) = \max_{i \in \mathcal{I}(t)} \dot{x}_i(t) = \max_{i \in \mathcal{I}(t)} \left[\sum_{j \in N_i(\sigma(t))} a_{ij}(t)(x_j(t) - x_i(t)) + w_i(t) \right] \leq \max_{i \in \mathcal{I}(t)} w_i(t) \leq |w(t)|,$$

which completes the proof. \square

We next establish two lemmas on UQSC graphs.

Lemma 3 *Suppose $\mathcal{G}_{\sigma(t)}$ is UQSC. Then there exists a center i_0 from which there is a path $i_0 \rightarrow i$ for all $i \in \mathcal{V}$ in $\mathcal{G}([t, t + \hat{T}])$ with $\hat{T} \doteq T + 2\tau_D$, and each arc of $i_0 \rightarrow i$ exists in a time interval with length τ_D at least during $[t, t + \hat{T})$.*

Proof. Denote t_1 as the first moment when the interaction topology switches within $[t, t + \hat{T})$ (to suppose there are switchings is without loss of generality).

If $t_1 \geq t + \tau_D$, then, there exists a center i_0 from which there is a path $i_0 \rightarrow i$ for all $i \in \mathcal{V}$ in $\mathcal{G}([t, t + T])$ since $\mathcal{G}([t, t + T])$ is quasi-strongly connected, and moreover, each arc of path $i_0 \rightarrow i$ stays there for at least the dwell time τ_D during $[t, t + T + \tau_D)$ due to the definition of τ_D .

On the other hand, if $t_1 < t + \tau_D$, we have $t_1 + T + \tau_D < t + \hat{T}$. Then, for any $i \in \mathcal{V}$, there is also a center i_0 from which there is a path $i_0 \rightarrow i$ for all $i \in \mathcal{V}$ in $\mathcal{G}([t_1, t_1 + T])$, each arc of which exists for at least τ_D during $[t_1, t_1 + T + \tau_D)$. This completes the proof. \square

Suppose that $\mathcal{G}_{\sigma(t)}$ is UQSC. Define a set-valued function $f : \mathbb{Z}^+ \rightarrow 2^{\{1, \dots, N\}}$, where $2^{\{1, \dots, N\}}$ represents the (power) set containing all the subsets of $\{1, \dots, N\}$:

$$f(s) = \{j \mid j \text{ is a center in } \mathcal{G}([(s-1)\hat{T}, s\hat{T}]) \text{ satisfying the condition of Lemma 3}\}, \quad s = 1, 2, \dots$$

Lemma 4 *Assume that $\mathcal{G}_{\sigma(t)}$ is UQSC and let d_0 be the (generalized) diameter of $\mathcal{G}([0, +\infty))$. Then for any $t = 1, 2, \dots$, there exists $k_0 \in \{1, 2, \dots, N\}$ such that $k_0 \in f(s)$ for s as many as at least d_0 during $s \in [t, t + (d_0 - 1)N]$.*

Proof. Suppose $k \in f(s)$ for less than d_0 times (i.e., less than or equal $d_0 - 1$) during $[t, t + (d_0 - 1)N]$ for all $k \in \{1, 2, \dots, N\}$. Then, the total number of the elements of all the preimages of f on interval $s \in [t, t + (d_0 - 1)N]$ is no larger than $(d_0 - 1)N$. However, on the other hand, there are at least $(d_0 - 1)N + 1$ elements (counting times for the same node) belonging to $f(s)$ during $s \in [t, t + (d_0 - 1)N]$ since $f(s) \neq \emptyset$ for all $s = 1, 2, \dots$. Then we get the contradiction and the conclusion is proved. \square

4.2 Proof of Theorem 1: UQSC Implies GRC

We are now in a position to prove the sufficiency statement in Theorem 1. Assume that the initial time is $t_0 = 0$ for simplicity. The analysis of $\mathcal{H}(x(t))$ will be carried out on time intervals $t \in [sK_0, (s+1)K_0]$ for $s = 0, 1, 2, \dots$, where $K_0 = [(d_0 - 1)N + 1]\hat{T}$.

Based on Lemma 2, we see that for all $t \in [sK_0, (s+1)K_0]$,

$$\hbar(t) \leq \hbar(sK_0) + \|w\|_\infty K_0; \quad \ell(t) \geq \ell(sK_0) - \|w\|_\infty K_0. \quad (5)$$

We divide the rest of the proof into three steps, in which convergence bound will be given over the network node by node on time intervals $[sK_0, (s+1)K_0]$, $s = 0, 1, \dots$. Assume that

$$x_{k_0}(sK_0) \leq \frac{1}{2}\ell(sK_0) + \frac{1}{2}\hbar(sK_0). \quad (6)$$

Step 1. According to Proposition 4, we can choose $k_0 \in \mathcal{V}$ be the center of d_0 joint graphs, $[j_m \hat{T}, (j_m + 1)\hat{T}] \subseteq [sK_0, (s+1)K_0]$, $m = 1, 2, \dots, d_0$. In this step, we bound $x_{k_0}(t)$ on time interval $[sK_0, (s+1)K_0]$.

With (5), we have

$$\frac{d}{dt}x_{k_0}(t) \leq -\mathcal{Y}_{k_0}(t)\left(x_{k_0}(t) - \hbar(sK_0) - K_0\|w\|_\infty\right) + |w(t)|, \quad t \in [sK_0, (s+1)K_0] \quad (7)$$

where $\mathcal{Y}_i(t) = \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(t)$, which implies

$$\begin{aligned} x_{k_0}(t) &\leq \left[1 - e^{-\int_{sK_0}^t \mathcal{Y}_{k_0}(\tau) d\tau}\right] (\hbar(sK_0) + K_0\|w\|_\infty) + e^{-\int_{sK_0}^t \mathcal{Y}_{k_0}(\tau) d\tau} x_{k_0}(sK_0) + K_0\|w\|_\infty \\ &\leq \xi_0 \ell(sK_0) + (1 - \xi_0) \hbar(sK_0) + 2K_0\|w\|_\infty, \quad t \in [sK_0, (s+1)K_0] \end{aligned} \quad (8)$$

where $\xi_0 = e^{-\lceil \frac{K_0}{\tau_D} \rceil a^*(N-1)}/2$ with $\lceil z \rceil$ denoting the smallest integer which is no smaller than z . Here the first inequality of (8) follows from Gronwall's inequality, and the second one holds based on assumption A2, (6) and the simple fact that $\ell(t) \leq \hbar(t)$.

Step 2. Since k_0 is a center in $\mathcal{G}([j_1 \hat{T}, (j_1 + 1)\hat{T}])$, we can well define a set $\mathcal{V}_1 = \{j : \exists t_1 \text{ s.t. } (k_0, j) \in \mathcal{G}_{\sigma(t)} \text{ for } t \in [t_1, t_1 + \tau_D] \subseteq [j_1 \hat{T}, (j_1 + 1)\hat{T}]\}$. In this step, we will establish an upper bound for $x_{k_1}(t)$, $k_1 \in \mathcal{V}_1$.

We have

$$\begin{aligned} \frac{d}{dt}x_{k_1}(t) &\leq \hat{\mathcal{Y}}_{k_1}(t) (\hbar(sK_0) + K_0\|w\|_\infty - x_{k_1}(t)) \\ &\quad + a_{k_1 k_0}(t) \left(\xi_0 \ell(sK_0) + (1 - \xi_0) \hbar(sK_0) + 2K_0\|w\|_\infty - x_{k_1}(t) \right) + w_{k_1}(t) \end{aligned}$$

for $t \in [t_1, t_1 + \tau_D]$, where $\hat{\mathcal{Y}}_{k_1}(t) = \mathcal{Y}_{k_1}(t) - a_{k_1 k_0}(t)$. Using Gronwall's inequality we thus obtain

$$\begin{aligned} x_{k_1}(t_1 + \tau_D) &\leq e^{-\int_{t_1}^{t_1 + \tau_D} \mathcal{Y}_{k_1}(t) dt} x_{k_1}(t_1) + (\hbar(sK_0) + K_0\|w\|_\infty) \int_{t_1}^{t_1 + \tau_D} e^{-\int_t^{t_1 + \tau_D} \mathcal{Y}_{k_1}(\tau) d\tau} \hat{\mathcal{Y}}_{k_1}(t) dt \\ &\quad + (\xi_0 \ell(sK_0) + (1 - \xi_0) \hbar(sK_0) + 2K_0\|w\|_\infty) \\ &\quad \cdot \int_{t_1}^{t_1 + \tau_D} e^{-\int_t^{t_1 + \tau_D} \mathcal{Y}_{k_1}(\tau) d\tau} a_{k_1 k_0}(t) dt + \tau_D \|w\|_\infty \\ &\leq \left(\xi_0 \int_{t_1}^{t_1 + \tau_D} e^{-\int_t^{t_1 + \tau_D} \mathcal{Y}_{k_1}(\tau) d\tau} a_{k_1 k_0}(t) dt \right) \ell(sK_0) \\ &\quad + \left(1 - \xi_0 \int_{t_1}^{t_1 + \tau_D} e^{-\int_t^{t_1 + \tau_D} \mathcal{Y}_{k_1}(\tau) d\tau} a_{k_1 k_0}(t) dt \right) \hbar(sK_0) + (2K_0 + \tau_D) \|w\|_\infty, \end{aligned} \quad (9)$$

where the second inequality follows from the facts that $x_{k_1}(t_1) \leq \hbar(sK_0) + K_0\|w\|_\infty$ and

$$\int_{t_1}^{t_1+\tau_D} e^{-\int_t^{t_1+\tau_D} \mathcal{Y}_{k_1}(\tau) d\tau} \mathcal{Y}_{k_1}(t) dt = 1 - e^{-\int_{t_1}^{t_1+\tau_D} \mathcal{Y}_{k_1}(t) dt}.$$

Furthermore, noticing that

$$\begin{aligned} \int_{t_1}^{t_1+\tau_D} e^{-\int_t^{t_1+\tau_D} \mathcal{Y}_{k_1}(\tau) d\tau} a_{k_1 k_0}(t) dt &= \int_{t_1}^{t_1+\tau_D} e^{-\int_t^{t_1+\tau_D} \hat{\mathcal{Y}}_{k_1}(\tau) d\tau} \cdot e^{-\int_t^{t_1+\tau_D} a_{k_1 k_0}(\tau) d\tau} a_{k_1 k_0}(t) dt \\ &\geq e^{-(N-2)a^*} \int_{t_1}^{t_1+\tau_D} e^{-\int_t^{t_1+\tau_D} a_{k_1 k_0}(\tau) d\tau} a_{k_1 k_0}(t) dt \\ &= e^{-(N-2)a^*} \left(1 - e^{-\int_{t_1}^{t_1+\tau_D} a_{k_1 k_0}(t) dt} \right) \\ &\geq e^{-(N-2)a^*} (1 - e^{-a^*}), \end{aligned} \quad (10)$$

we conclude from (9) that

$$x_{k_1}(t_1 + \tau_D) \leq \zeta \xi_0 \ell(sK_0) + (1 - \zeta \xi_0) \hbar(sK_0) + (2K_0 + \tau_D) \|w\|_\infty, \quad (11)$$

where $\zeta = e^{-(N-2)a^*} (1 - e^{-a^*})$.

Therefore, (11) implies that for any $k_1 \in \mathcal{V}_1$, there exists an instance $t_* \in [j_1 \hat{T}, (j_1 + 1) \hat{T})$ such that

$$x_{k_1}(t_*) \leq \zeta \xi_0 \ell(sK_0) + (1 - \zeta \xi_0) \hbar(sK_0) + 3K_0 \|w\|_\infty. \quad (12)$$

Applying inequality (7) on x_{k_1} for $t \in [t_*, (s+1)K_0]$, it turns out that

$$x_{k_1}(t) \leq \xi_1 \ell(sK_0) + (1 - \xi_1) \hbar(sK_0) + 4K_0 \|w\|_\infty, \quad t \in [(j_1 + 1) \hat{T}, (s+1)K_0] \quad (13)$$

for all $k_1 \in \mathcal{V}_1$, where $\xi_1 = e^{-\lceil \frac{K_0}{\tau_D} \rceil a^* (N-1)} e^{-(N-2)a^*} (1 - e^{-a^*}) \cdot \xi_0$.

Step 3. Continuing the analysis on time interval $[j_2 \hat{T}, (j_2 + 1) \hat{T})$, we can similarly define $\mathcal{V}_2 = \{j : \exists t_2 \text{ s.t. there is an arc from } \{k_0\} \cup \mathcal{V}_1 \text{ to } j \text{ for } t \in [t_2, t_2 + \tau_D] \subseteq [j_2 \hat{T}, (j_2 + 1) \hat{T})\}$. Repeating the analysis in Step 2, we have

$$x_{k_2}(t) \leq \xi_2 \ell(sK_0) + (1 - \xi_2) \hbar(sK_0) + 8K_0 \|w\|_\infty, \quad t \in [(j_2 + 1) \hat{T}, (s+1)K_0] \quad (14)$$

for all $k_2 \in \mathcal{V}_2$, where $\xi_2 = e^{-\lceil \frac{K_0}{\tau_D} \rceil a^* (N-1)} e^{-(N-2)a^*} (1 - e^{-a^*}) \cdot \xi_1$.

Recall that d_0 is the (generalized) diameter of $\mathcal{G}([0, +\infty))$. We can proceed the analysis on time intervals $[j_m \hat{T}, (j_{m+1}) \hat{T})$ for $m = 3, \dots, d_0$ until we obtain

$$x_i((s+1)K_0) \leq \xi_{d_0} \ell(sK_0) + (1 - \xi_{d_0}) \hbar(sK_0) + 4d_0 K_0 \|w\|_\infty, \quad i = 1, \dots, N \quad (15)$$

where

$$\xi_{d_0} = e^{-\lceil \frac{K_0}{\tau_D} \rceil a^* (d_0+1)(N-1)} e^{-(N-2)d_0 a^*} (1 - e^{-a^*})^{d_0} / 2. \quad (16)$$

This leads to

$$\begin{aligned} \mathcal{H}(x((s+1)K_0)) &\leq \xi_{d_0} \ell(sK_0) + (1 - \xi_{d_0}) \hbar(sK_0) + 4d_0 K_0 \|w\|_\infty - (\ell(sK_0) - \|w\|_\infty K_0) \\ &= (1 - \xi_{d_0}) \mathcal{H}(x(sK_0)) + (4d_0 + 1) K_0 \|w\|_\infty. \end{aligned} \quad (17)$$

For the opposite case of (6) with $x_{k_0}(sK_0) > \frac{1}{2}\ell(sK_0) + \frac{1}{2}h(sK_0)$, we see that (17) also holds using a symmetric argument by investigating the lower bound for $\ell((s+1)K_0)$.

Since s is arbitrarily chosen in (17), we have

$$\begin{aligned}\mathcal{H}(x(nK_0)) &\leq (1 - \xi_{d_0})^n \mathcal{H}(x^0) + \sum_{j=0}^{n-1} (1 - \xi_{d_0})^j (4d_0 + 1) K_0 \|w\|_\infty \\ &\leq (1 - \xi_{d_0})^n \mathcal{H}(x^0) + \frac{(4d_0 + 1) K_0}{\xi_{d_0}} \cdot \|w\|_\infty\end{aligned}$$

for any $n = 0, 1, 2, \dots$. From (5), we also know

$$\mathcal{H}(x(t)) \leq \mathcal{H}(x(nK_0)) + 2K_0 \|w\|_\infty, \quad t \in [nK_0, (n+1)K_0]. \quad (18)$$

The desired GRC inequality is therefore obtained by

$$\beta(\mathcal{H}(x^0), t) = (1 - \xi_{d_0})^{\lfloor \frac{t}{K_0} \rfloor} \mathcal{H}(x^0), \quad \gamma(\|w\|_\infty) = (2 + \frac{4d_0 + 1}{\xi_{d_0}}) K_0 \cdot \|w\|_\infty, \quad (19)$$

where $\lfloor \frac{t}{K_0} \rfloor$ denotes the largest integer no greater than $\frac{t}{K_0}$. The proof is completed.

4.3 Convergence Time

The concept of ϵ -convergence time, sometimes also called ϵ -computation or ϵ -averaging time, has been introduced for discrete-time consensus algorithms to describe the required steps for the network to reach a certain level of consensus captured by a parameter ϵ , and bounds of this ϵ -convergence time have been extensively established in the literature [28, 16, 30, 31].

Now let us introduce the following definition of convergence time for the corresponding continuous-time version (2) in the absence of noise. Suppose $w(t) \equiv 0$. We define

$$T_N(\epsilon) = \sup_{x^0 \in \mathbb{R}^N, \mathcal{H}(x^0) \neq 0} \min \left\{ t : \frac{\mathcal{H}(x(t))}{\mathcal{H}(x^0)} \leq \epsilon \right\}. \quad (20)$$

From (19), we know that under UQSC communication graphs,

$$\mathcal{H}(x(t)) \leq (1 - \xi_{d_0})^{\lfloor \frac{t}{K_0} \rfloor} \mathcal{H}(x^0) \leq (1 - \xi_{d_0})^{\frac{t}{K_0} - 1} \mathcal{H}(x^0) = \frac{1}{1 - \xi_{d_0}} e^{-\lambda_0 t} \mathcal{H}(x^0). \quad (21)$$

where

$$\lambda_0 = \frac{1}{K_0} \ln \frac{1}{1 - \xi_{d_0}}. \quad (22)$$

Hence, simple computation leads to

$$T_N(\epsilon) \leq \frac{\log((1 - \xi_{d_0})\epsilon)^{-1}}{\lambda_0} = O\left(N e^{\left(\lceil \frac{K_0}{\tau_D} \rceil a^*(d_0+1) + d_0 a^*\right)N} \log \epsilon^{-1}\right) \quad (23)$$

where by definition $a_N = O(b_N)$ means that $\lim_{N \rightarrow \infty} \frac{a_N}{b_N}$ is a nonzero constant.

On the other hand, if $\mathcal{G}_{\sigma(t)}$ is USC, for any $t \geq 0$ and any node $k \in \mathcal{V}$, k will be the center of joint graphs on $N - 1$ subintervals $[t, t + \hat{T}), \dots, [t + (N - 2)\hat{T}, t + (N - 1)\hat{T})$. Moreover, the generalized diameter of $\mathcal{G}([0, +\infty))$ is exactly $N - 1$ for USC graphs. Therefore, replacing K_0 with $K_* = (N - 1)\hat{T}$ and based on the same analysis as in the proof of Theorem 1, similar GRC inequality under USC graphs can be given by

$$\beta(\mathcal{H}(x^0), t) = (1 - \xi_{d_0}^*)^{\lfloor \frac{t}{K_*} \rfloor} \mathcal{H}(x^0), \quad \gamma(\|w\|_\infty) = (2K_* + \frac{4N - 3}{\xi_{d_0}^*}) \cdot \|w\|_\infty,$$

where

$$\xi_{d_0}^* = e^{-\lceil \frac{K_*}{\tau_D} \rceil a^* N(N-1)} e^{-(N-2)(N-1)a^*} (1 - e^{-a_*})^{N-1} / 2.$$

Therefore, under USC communication graphs, we can similarly obtain

$$T_N(\epsilon) \leq O\left(N(1 - e^{-a_*})^{1-N} e^{(N-1)a^*} \left(\lceil \frac{K_*}{\tau_D} \rceil^{N+N-2}\right)\right) \log \epsilon^{-1}. \quad (24)$$

Remark 3 Compared to the results for discrete-time consensus dynamics with the same (USC) connectivity condition [30, 31], the convergence time given in (24) is relatively conservative. Intuitively, the system should achieve faster convergence when the topology is USC compared to when it is UQSC. However, this point is not captured in (23) and (24). The reason for this is that the approach taken in the proof of Theorem 1 is targeting particularly UQSC graphs, though it can also be used for USC graphs. We believe that there exist sharper bounds for the convergence time under continuous-time dynamics.

4.4 L^∞ -Vanishing Noise

Consider a set defined by

$$\mathcal{F}_1 \doteq \{z \in \mathcal{F} : \lim_{t \rightarrow \infty} z(t) = 0\},$$

and let $\mathcal{F}_1^0 \subseteq \mathcal{F}_1$ be a subset with $\lim_{t \rightarrow \infty} \sup_{z \in \mathcal{F}_1^0} |z(t)| = 0$. Then the following corollary holds.

Proposition 1 (i) System (2) achieves a GC for any $w \in \mathcal{F}_1$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

(ii) System (2) achieves a GAC with respect to \mathcal{F}_1^0 iff $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proof. (i) Suppose β and γ are defined as (19). Let $w_0 \in \mathcal{F}_1$ be a fixed function. Then, $\forall \epsilon > 0$, $\exists T(\epsilon) > 0$ such that $|w_0(t)| < \gamma^{-1}(\epsilon)$, $\forall t \geq T(\epsilon)$. Thus, applying Theorem 1 on system (2) with $t_0 = T(\epsilon)$, we obtain

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x(T(\epsilon))), t - T(\epsilon)) + \epsilon. \quad (25)$$

Since ϵ can be arbitrarily small, the global consensus follows immediately by taking $t \rightarrow \infty$ in (25).

(ii) (Sufficiency.) Suppose β and γ are defined as (19). Then $\forall \epsilon > 0$, $\exists \tilde{T}(\epsilon) > 0$ such that $|w(t)| \leq \gamma^{-1}(\frac{\epsilon}{2})$, $\forall t \geq \tilde{T}(\epsilon)$, $\forall w \in \mathcal{F}_1^0$. Denoting $\omega^* = \sup_{t \in [t_0, \tilde{T}]} \{\sup_{z \in \mathcal{F}_1^0} |z(t)|\}$, there will be two cases.

- When $t_0 < \tilde{T}(\varepsilon)$, one has $\forall t \geq t_0$,

$$\begin{aligned}\mathcal{H}(x(t)) &\leq \beta\left(\mathcal{H}(x(\tilde{T}(\varepsilon))), t - \tilde{T}(\varepsilon)\right) + \frac{\varepsilon}{2} \\ &\leq \beta\left(\beta(\mathcal{H}(x^0) + \gamma(\omega^*), \tilde{T}(\varepsilon) - t_0), t - \tilde{T}(\varepsilon)\right) + \frac{\varepsilon}{2} \\ &\leq \beta\left(\beta(\mathcal{H}(x^0) + \gamma(\omega^*), 0), t - \tilde{T}(\varepsilon)\right) + \frac{\varepsilon}{2}.\end{aligned}\tag{26}$$

Furthermore, $\forall c > 0$, $\exists T_1(c, \tilde{T}(\varepsilon)) > 0$ such that

$$\beta\left(\beta(c + \gamma(\omega^*), 0), t - \tilde{T}(\varepsilon)\right) \leq \frac{\varepsilon}{2}, \forall t \geq T_1,$$

- When $t_0 \geq \tilde{T}(\varepsilon)$, one has $\forall t \geq t_0$,

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x^0), t - t_0) + \frac{\varepsilon}{2}.\tag{27}$$

Then $\forall c > 0$, $\exists T_2(c) > 0$ such that $\beta(\mathcal{H}(x^0), t - t_0) \leq \frac{\varepsilon}{2}, \forall t \geq T_2$.

Taking $T = \max\{T_1, T_2\}$, we obtain

$$\mathcal{H}(x^0) \leq c \Rightarrow \mathcal{H}(x(t)) \leq \varepsilon, \forall t \geq t_0 + T, \forall w \in \mathcal{F}_1^0.$$

Hence the sufficient part is proved.

(Necessity.) Suppose $\mathcal{G}_{\sigma(t)}$ is not UQSC. Then $\forall \varepsilon > 0, \forall T_* > 0, \exists W > 0$, such that $|w(t)| \leq \frac{\varepsilon}{2T_*}, \forall t \geq W$. Moreover, $\forall T_* > 0, \exists t_* > M$ such that $\mathcal{G}([t_*, t_* + T_*])$ is not quasi-strongly connected. Furthermore, we define \mathcal{V}_1 and \mathcal{V}_2 in the same way as the proof of Theorem 1. Let initial condition be $t_0 = t_*$ with $x_i(t_*) = 0, \forall i \in \mathcal{V}_1$ and $x_i(t_*) = c, \forall i \in \mathcal{V}_2$. Then it is not hard to find that $\mathcal{H}(x(t_* + T_*)) \geq c - \varepsilon$. Therefore, the global asymptotic consensus cannot be achieved since T_* can be arbitrarily large. \square

4.5 UQSC and GIRC

4.5.1 Non-conservativeness

Theorem 2 only states the sufficiency of UQSC graph for GIRC. Let us see the following simple example which shows that the corresponding necessity claim does not hold.

Example 1 Suppose there are only two nodes, 1 and 2, in the network. The arc set is $\mathcal{E}_{\sigma(t)} = \{(1, 2)\}$ if $t \in [10^k, 10^k + 1)$ for $k = 0, 1, \dots$, and $\mathcal{E}_{\sigma(t)} = \emptyset$ otherwise. Take $a_{12}(t) \equiv a_{21}(t) \equiv 1$. Then we have

$$\frac{d}{dt}(x_1(t) - x_2(t)) = \begin{cases} -(x_1(t) - x_2(t)) + w_1(t) - w_2(t), & t \in [10^k, 10^k + 1), k = 0, 1, \dots \\ w_1(t) - w_2(t), & \text{otherwise} \end{cases}\tag{28}$$

It is not hard to see that the system in Example 1 achieves a GIRC, though the varying communication graph is not UQSC. Therefore, UQSC graph is no longer necessary to ensure GIRC with general directed communications.

4.5.2 Proof of Theorem 2

The proof follows the same line as the proof of Theorem 1. We will bound $\mathcal{H}(x(t))$ on time intervals $t \in [sK_0, (s+1)K_0]$ for $s = 0, 1, \dots$. Denote $\eta_s = \int_{sK_0}^{(s+1)K_0} |w(t)| dt$. Then based on Lemma 2, for any $t \in [sK_0, (s+1)K_0]$, we have

$$x_i(t) \in [\ell(s\hat{T}) - \eta_s, \hbar(s\hat{T}) + \eta_s], \quad i = 1, \dots, N. \quad (29)$$

Suppose k_0 is a node as defined in the proof of Theorem 1. Provided, without loss of generality, that $x_{k_0}(sK_0) \leq \frac{1}{2}\ell(sK_0) + \frac{1}{2}\hbar(sK_0)$ and as that

$$\frac{d}{dt}x_{k_0}(t) \leq -\mathcal{Y}_{k_0}(t) \left(x_{k_0}(t) - \hbar(sK_0) - \eta_s \right) + |w(t)|, \quad t \in [sK_0, (s+1)K_0] \quad (30)$$

which implies

$$\begin{aligned} x_{k_0}(t) &\leq \left[1 - e^{-\int_{sK_0}^t \mathcal{Y}_{k_0}(\tau) d\tau} \right] (\hbar(sK_0) + \eta_s) + e^{-\int_{sK_0}^t \mathcal{Y}_{k_0}(\tau) d\tau} x_{k_0}(sK_0) + \int_{sK_0}^t e^{-\int_{sK_0}^z \mathcal{Y}_{k_0}(\tau) d\tau} |w(z)| dz \\ &\leq \xi_0 \ell(sK_0) + (1 - \xi_0) \hbar(sK_0) + 2\eta_s, \quad t \in [sK_0, (s+1)K_0] \end{aligned} \quad (31)$$

where the second inequality follows from the simple fact that $0 < e^{-\int_{sK_0}^t \mathcal{Y}_{k_0}(\tau) d\tau} \leq 1$.

Therefore, similar to the proof of Theorem 1, the analysis can be carried on node by node for different disjoint intervals and we can eventually arrive at

$$\mathcal{H}(x((s+1)K_0)) \leq (1 - \xi_{d_0}) \mathcal{H}(x(sK_0)) + (4d_0 + 1)\eta_s. \quad (32)$$

Consequently, for any $n = 0, 1, 2, \dots$, it holds that

$$\begin{aligned} \mathcal{H}(x(nK_0)) &\leq (1 - \xi_{d_0})^n \mathcal{H}(x^0) + (4d_0 + 1) \sum_{j=0}^{n-1} (1 - \xi_{d_0})^{n-1-j} \eta_j \\ &\leq (1 - \xi_{d_0})^n \mathcal{H}(x^0) + (4d_0 + 1) \sum_{j=0}^{n-1} \eta_j \end{aligned} \quad (33)$$

Thus, together with the observation that

$$\mathcal{H}(x(t)) \leq \mathcal{H}(x(nK_0)) + \int_{nK_0}^t |w(\tau)| d\tau, \quad t \in [nK_0, (n+1)K_0], \quad (34)$$

the following GIRC inequality is obtained:

$$\mathcal{H}(x(t)) \leq (1 - \xi_{d_0})^{\lfloor \frac{t}{K_0} \rfloor} \mathcal{H}(x^0) + (4d_0 + 1) \int_0^t |w(\tau)| d\tau. \quad (35)$$

This completes the proof. \square

Remark 4 A sharper inequality can be obtained based on the first inequality of (33) that

$$\mathcal{H}(x(t)) \leq (1 - \xi_{d_0})^{\lfloor \frac{t}{K_0} \rfloor} \mathcal{H}(x^0) + (4d_0 + 1) \int_0^t (1 - \xi_{d_0})^{\lfloor \frac{t}{K_0} \rfloor - \lceil \frac{\tau}{K_0} \rceil} |w(\tau)| d\tau, \quad (36)$$

which will be useful in the following discussions on distributed event-triggered consensus.

5 Convergence: Bidirectional Graphs

This section focuses on the proof of Theorem 3. In what follows of this section, we assume that the communications over the network is bidirectional, i.e., $\mathcal{G}_{\sigma(t)}$ is bidirectional graph for all $t \geq 0$.

5.1 Time-axis Partition

We introduce a partition, $0 = T_0 < T_1 < T_2 < \dots$, for the time-axis.

Let $T_0 = 0$. Then $T_k, k = 1, 2, \dots$, can be defined by induction as

$$T_k = \inf \left\{ t \geq T_{k-1} : \mathcal{G}([T_{k-1}, t]) \text{ has a connected spanning subgraph,} \right. \\ \left. \text{each arc of which exists at least } \tau_D \text{ time within time interval } [T_{k-1}, t) \right\}.$$

See that when $\mathcal{G}_{\sigma(t)}$ is IJC, T_k is finite for any fixed $k = 1, 2, \dots$.

We can further define

$$J(t) = \max\{k : t > T_k\}.$$

Then $J(t)$ characterizes how many times for different proper joint graphs being jointly connected during time interval $[0, t)$.

5.2 Proof of Theorem 3

The necessity part of the conclusion is straightforward, so we just focus on the sufficient part.

Denote $\varpi_0 = \int_{T_0}^{T_{d_0}} |w(t)| dt$. Then based on Lemma 2, we have

$$\bar{h}(t) \leq \bar{h}(T_0) + \varpi_0; \quad \ell(t) \geq \ell(T_0) - \varpi_0 \quad (37)$$

for all $T_0 \leq t \leq T_{d_0}$.

We divide the following analysis into five steps.

Step 1. Take $i_0 \in \mathcal{V}$ with $x_{i_0}(T_0) = \ell(T_0)$. Denote \bar{t}_1 by

$$\bar{t}_1 \doteq \inf \{ t \geq T_0 : \text{at least one different node connects } i_0 \text{ in } \mathcal{G}_{\sigma(t)} \}.$$

Since $\mathcal{G}_{\sigma(t)}$ is infinitely jointly connected, we have that $\bar{t}_1 + \tau_D \leq T_1$ according to the definition of T_1 .

Note that, we have

$$x_{i_0}(\bar{t}_1) \leq \ell(T_0) + \int_{T_0}^{\bar{t}_1} |w(t)| dt,$$

because no other node is connected to i_0 during $[T_0, \bar{t}_1]$. Then similar to (8), the following inequality

$$\frac{d}{dt} x_{i_0}(t) \leq -\mathcal{Y}_{i_0}(t) \left(x_{i_0}(t) - \bar{h}(T_0) - \varpi_0 \right) + |w(t)|, \quad t \in [T_0, T_{d_0}] \quad (38)$$

implies

$$\begin{aligned}
x_{i_0}(t) &\leq \left[1 - e^{-\int_{\bar{t}_1}^t \mathcal{Y}_{i_0}(\tau) d\tau}\right] (\bar{h}(T_0) + \varpi_0) + e^{-\int_{\bar{t}_1}^t \mathcal{Y}_{i_0}(\tau) d\tau} x_{i_0}(\bar{t}_1) + \int_{\bar{t}_1}^{\bar{t}_1 + \tau_D} |w(t)| dt \\
&\leq e^{-\int_{\bar{t}_1}^t \mathcal{Y}_{i_0}(\tau) d\tau} \ell(\bar{t}_1) + \left[1 - e^{-\int_{\bar{t}_1}^t \mathcal{Y}_{i_0}(\tau) d\tau}\right] \bar{h}(T_0) + \varpi_0 + \int_{T_0}^{\bar{t}_1} |w(t)| dt + \int_{\bar{t}_1}^{T_{d_0}} |w(t)| dt \\
&\leq m_0 \ell(sK_0) + (1 - m_0) \bar{h}(sK_0) + 2\varpi_0
\end{aligned} \tag{39}$$

for all $t \in [T_0, \bar{t}_1 + \tau_D]$, where $m_0 = e^{-(N-1)a^*}$.

Step 2. Denote $\bar{\mathcal{V}}_1 \doteq \{j : j \text{ is a neighbor of } i_0 \text{ in } \mathcal{G}([\bar{t}_1, \bar{t}_1 + \tau_D])\}$. In this step, we bound $x_{i_1}(t)$ for $i_1 \in \bar{\mathcal{V}}_1$ in time interval $[\bar{t}_1, \bar{t}_1 + \tau_D]$.

Based on similar analysis with (11), we can conclude from (39) that

$$x_{i_1}(\bar{t}_1 + \tau_D) \leq \zeta m_0 \ell(T_0) + (1 - \zeta m_0) \bar{h}(T_0) + 2\varpi_0 + \int_{\bar{t}_1}^{\bar{t}_1 + \tau_D} |w(\tau)| d\tau, \tag{40}$$

where $\zeta = e^{-(N-2)a^*} (1 - e^{-a^*})$.

Step 3. In this step, let us further discuss the bound of $x_i(t)$ after $\bar{t}_1 + \tau_D$ for nodes in $\{i_0\} \cup \bar{\mathcal{V}}_1$.

Let us view T_k as a switching instance in the graph signal $\sigma(t)$ for any $k = 0, 1, \dots$. Denote p_1 as the first switching instance after \bar{t}_1 . Then $p_1 \leq T_1$. There will be two cases:

- (i) If $p_1 - \bar{t}_1 < 2\tau_D$, we can relax (39) and (40) on time interval $[T_0, p_1]$. In this case, we have for any $i \in \{i_0\} \cup \bar{\mathcal{V}}_1$,

$$x_i(p_1) \leq \eta_0 \ell(T_0) + (1 - \eta_0) \bar{h}(T_0) + 2\varpi_0 + \int_{\bar{t}_1}^{p_1} |w(\tau)| d\tau \tag{41}$$

where $\eta_0 = e^{-2(N-1)a^*} e^{-2(N-2)a^*} (1 - e^{-2a^*}) = e^{-2(2N-3)a^*} (1 - e^{-2a^*})$.

- (ii) If there is no other node connecting $\{i_0\} \cup \bar{\mathcal{V}}_1$ for $t \in [\bar{t}_1 + \tau_D, p]$, applying Lemma 2 on the subsystem formed by $\{i_0\} \cup \bar{\mathcal{V}}_1$, we obtain

$$\begin{aligned}
x_i(p) &\leq \zeta m_0 \ell(T_0) + (1 - \zeta m_0) \bar{h}(T_0) + 2\varpi_0 + \int_{T_0}^{\bar{t}_1 + \tau_D} |w(\tau)| d\tau + \int_{\bar{t}_1 + \tau_D}^p |w(\tau)| d\tau \\
&\leq \eta_0 \ell(T_0) + (1 - \eta_0) \bar{h}(T_0) + 2\varpi_0 + \int_{T_0}^p |w(\tau)| d\tau
\end{aligned} \tag{42}$$

for all $i \in \{i_0\} \cup \bar{\mathcal{V}}_1$.

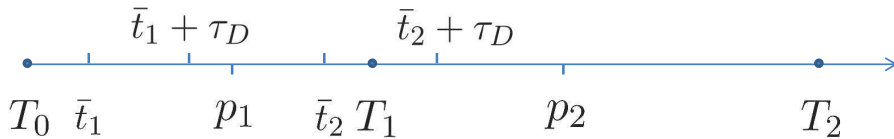


Figure 1: The structure of the proof.

Step 4. In this step, we continue to analyze the neighbors of $\{i_0\} \cup \bar{\mathcal{V}}_1$. Based on (41) and (42), if we define

$$\bar{t}_2 = \inf \{t \geq \bar{t}_1 + \tau_D : \text{at least one other node connects } \{i_0\} \cup \bar{\mathcal{V}}_1 \text{ in } \mathcal{G}_{\sigma(s)} \text{ for any } s \in [t, t + \tau_D)\},$$

and

$$\bar{\mathcal{V}}_2 \doteq \{j : j \text{ has a neighbor which belongs to } \{i_0\} \cup \bar{\mathcal{V}}_1 \text{ in } \mathcal{G}([\bar{t}_2, \bar{t}_2 + \tau_D))\},$$

then we have $\bar{t}_2 + \tau_D \leq T_2$ and

$$x_i(\bar{t}_2) \leq \eta_0 \ell(T_0) + (1 - \eta_0) \hbar(T_0) + 3\varpi_0, \quad i \in \{i_0\} \cup \bar{\mathcal{V}}_1. \quad (43)$$

Thus, similar to (40), a upper bound for any $i_2 \in \bar{\mathcal{V}}_2$ can thus be obtained by

$$x_{i_2}(\bar{t}_2 + \tau_D) \leq \zeta m_0 \eta_0 \ell(T_0) + (1 - \zeta m_0 \eta_0) \hbar(T_0) + 5\varpi_0 + \int_{\bar{t}_2}^{\bar{t}_2 + \tau_D} |w(\tau)| d\tau, \quad (44)$$

and (44) also holds for node in $\{i_0\} \cup \bar{\mathcal{V}}_1$. Moreover, if $p_2 - \bar{t}_2 < 2\tau_D$, where p_2 is the first switching instance after $\bar{t}_2 + \tau_D$, a relaxed bound can also be obtained as

$$x_i(p_2) \leq \eta_0^2 \ell(T_0) + (1 - \eta_0^2) \hbar(T_0) + 6\varpi_0 \quad (45)$$

for all $i \in \{i_0\} \cup \bar{\mathcal{V}}_1 \cup \bar{\mathcal{V}}_2$.

Step 5. Proceeding this analysis, $\bar{t}_3, \dots, \bar{t}_{d_0}$ can be found respectively, and we eventually have

$$x_i(\bar{t}_{d_0} + \tau_D) \leq \eta_0^{d_0} \ell(T_s) + (1 - \eta_0^{d_0}) \hbar(T_s) + 3d_0 \varpi_0 \quad (46)$$

where $\bar{t}_{d_0} + \tau_D \leq T_{d_0}$, which implies

$$\mathcal{H}(x(T_{d_0})) \leq (1 - \eta_0^{d_0}) \mathcal{H}(x(T_0)) + (3d_0 + 1) \varpi_0. \quad (47)$$

Since (47) holds independent with the initial condition, we can further conclude that

$$\mathcal{H}(x(T_{nd_0})) \leq (1 - \eta_0^{d_0})^n \mathcal{H}(x^0) + (3d_0 + 1) \sum_{j=0}^{n-1} (1 - \eta_0^{d_0})^{n-1-j} \varpi_j \quad (48)$$

for all $n = 0, 1, 2, \dots$, where $\varpi_j = \int_{T_{jd_0}}^{T_{(j+1)d_0}} |w(t)| dt$.

Therefore, the desired GIRC inequality can be obtained by

$$\mathcal{H}(x(t)) \leq (1 - \eta_0^{d_0})^{\lfloor \frac{J(t)}{d_0} \rfloor} \mathcal{H}(x^0) + (3d_0 + 1) \int_0^t (1 - \eta_0^{d_0})^{\lfloor \frac{J(t)}{d_0} \rfloor - \lceil \frac{J(\tau)}{d_0} \rceil} |w(\tau)| d\tau \quad (49)$$

The proof is completed. \square

5.3 Convergence Time: “Exponential” Convergence

Suppose $w(t) \equiv 0$. Then we see from (49) that

$$\mathcal{H}(x(t)) \leq (1 - \eta_*)^{\lfloor \frac{J(t)}{d_0} \rfloor} \mathcal{H}(x^0) \leq (1 - \eta_*)^{-1} e^{-d_0^{-1} \log(1 - \eta_*)^{-1} J(t)} \mathcal{H}(x^0). \quad (50)$$

where

$$\eta_* \doteq \eta_0^{d_0} = e^{-(4N-6)d_0 a^*} (1 - e^{-2a_*})^{d_0}.$$

Now we see from (50) that when the system topology $\mathcal{G}_{\sigma(t)}$ is IJC, system (2) with bidirectional communications will reach a consensus exponentially with respect to $J(t)$ in the absence of noise, which is the times of the joint-connection being connected.

Furthermore, an upper bound for the ϵ -convergence time $T_N(\epsilon)$ under IJC communication graph can be established by

$$\begin{aligned} T_N(\epsilon) &\leq \inf \left\{ t : J(t) \leq \frac{d_0}{\log(1 - \eta_*)^{-1}} \log(\epsilon(1 - \eta_*))^{-1} \right\} \\ &\leq \inf J^{-1} \left(\lceil O(e^{(4N-6)d_0 a^*}) \log \epsilon^{-1} \rceil \right) \end{aligned} \quad (51)$$

where $J^{-1}(z) = \{t : J(t) = z\}$.

5.4 L^1 -Vanishing Noise

Consider the following set:

$$\mathcal{F}_2 \doteq \left\{ z \in \mathcal{F} : \int_0^\infty |z(t)| dt < \infty \right\}.$$

Let $\mathcal{F}_2^0 \subseteq \mathcal{F}_2$ be a subset of \mathcal{F}_2 with $\int_0^\infty \sup_{z \in \mathcal{F}_2^0} |z(t)| dt < \infty$. Then the following corollary holds.

Proposition 2 (i) System (2) achieves a GAC with respect to \mathcal{F}_2^0 iff $\mathcal{G}_{\sigma(t)}$ is UQSC.

(ii) Assume that $\mathcal{G}_{\sigma(t)}$ is bidirectional for any $t \geq 0$. Then System (2) achieves a GC for all $w \in \mathcal{F}_2$ iff $\mathcal{G}_{\sigma(t)}$ is IJC.

This proposition follows straightforwardly from the GIRC property given in the previous section. The proof is therefore omitted.

Remark 5 The ideas to obtain Propositions 1 and 2 are from ISS and iISS properties which study the conditions to ensure the system state converge to zero [32]. Basically, the results show that to reach a GAC for system (2), the system topology has to be UQSC. This is consistent with the main result in [35], in which the problem is studied without disturbances. On the other hand, the results also show that if we require a simple consensus, $[t, \infty)$ -joint connectedness is enough with bidirectional communications.

6 Application: Distributed Event-triggered Consensus

Distributed event-triggered coordination for multi-agent systems means that the control input of each agent should be piecewise constant based on neighboring information [47, 48].

It has been shown that event-based control needs fewer samples than time-triggered control to achieve the same performance for stochastic systems [43]. Recently event-triggered control has attracted much research interest [45]. Up to distributed event-triggered coordination rules, the system may also benefit from reducing the communication frequency over the network.

We first define the time instances of event-triggered executions for each node. Denote $t_0^i < t_1^i < \dots < t_k^i < \dots$ as the time sequence when agent i is triggered. Let $t_0^i = 0$ be the initial time. Having got $t_k^i, i \geq 0$, we denote $e_i(t) \doteq x_i(t) - x_i(t_k^i)$ as the error function of node i . Let L_0 be a given constant and $\delta(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ be a given function. Then t_{k+1}^i is determined by the following triggering condition with forceful waking-up.

- (a) Node i keeps checking whether the following equation is satisfied:

$$|e_i(t)| = \delta(t); \quad (52)$$

- (b) Node i chooses $t_{k+1}^i = t_k^i + L_0$ if (52) has never been satisfied over time interval $[t_k^i, t_k^i + L_0]$. Otherwise, t_{k+1}^i equals the first time instance when (52) holds.

Remark 6 *If the triggering condition is totally determined by equation (52), a node will never be triggered again once its control input equals zero at some time t_k^i . Consequently, a global consensus cannot be guaranteed. This is why the forceful waking up (timeout) condition is introduced to the triggering condition.*

Next, the communication and updating protocol for the considered distributed event-triggered coordination control is stated as follows.

- (i) (*Broadcasting*) Each agent i broadcasts its state $x_i(t_k^i)$ during $[t_k^i, t_{k+1}^i)$ until it is triggered another time at t_{k+1}^i .
- (ii) (*Receiving*) Agent j can receive $x_i(t_k^i)$ if and only if there exists a time $t_1 \in [t_k^i, t_{k+1}^i)$ such that i is a neighbor of j at time t_1 . Moreover, agent j can store this message until another message from i is received.
- (iii) (*Updating*) Each agent i updates its control input at time $x_i(t_k^i)$ once it is triggered, based on the messages it receives from the neighbor set $\hat{\mathcal{N}}_i(k) \doteq \bigcup_{t \in [t_{k-1}^i, t_k^i)} \mathcal{N}_i(\sigma(t))$.

Sticking to rule (i)-(iii), we present the control rule for each node i :

$$u_i(t) = \sum_{j \in \hat{\mathcal{N}}_i(k)} \left(x_j(t_{\mathcal{T}_i^j(k)}^j) - x_i(t_k^i) \right), \quad t \in [t_k^i, t_{k+1}^i), \quad (53)$$

where $\mathcal{T}_i^j(k) \doteq \max_l \{l : t_l^j \leq T_{ij}^*(k)\}$ with $T_{ij}^*(k) \doteq \max_t \{t \in [t_{k-1}^i, t_k^i) : j \in \mathcal{N}_i(\sigma(t))\}$.

Denote $\mathcal{T}_i(t) = \arg \max_l \{t_l^i | t_l^i \leq t\}$, $i = 1, \dots, N$, and

$$\hat{w}_i(t) = \sum_{j \in \hat{\mathcal{N}}_i(k)} (e_i(t) - e_j(t)) + \sum_{j \in \hat{\mathcal{N}}_i(k)} (x_j(t_{\mathcal{T}_i^j(k)}^j) - x_j(t_{\mathcal{T}_j(t)}^j)). \quad (54)$$

Then we can write (53) into the following form:

$$u_i(t) = \sum_{j \in \hat{\mathcal{N}}_i(k)} (x_j(t) - x_i(t)) + \hat{w}_i(t). \quad (55)$$

Furthermore, denote $\tau_{k+1}^i = t_{k+1}^i - t_k^i$ for $k = 0, 1, \dots$ and $i = 1, \dots, N$ as the difference between two consecutive event time instances, and let $\tau_0 \doteq \min_i \inf_k \{\tau_{k+1}^i\}$ be their lower bound. Note that, if $\tau_0 > 0$, the *Zeno* behavior, which indicates infinite triggering in finite time [46], is then avoided.

Take

$$\xi_* = e^{-\lceil \frac{K_0}{\tau_D} \rceil (d_0+1)(N-1)} e^{-(N-2)d_0} (1 - e^{-1})^{d_0} / 2$$

as the case with $a_* = a^* = 1$ in the definition of ξ_{d_0} in (16). Denote

$$A_0 = \frac{1}{1 - \xi_*}, \quad \theta_0 = \frac{1}{K_0} \ln \frac{1}{1 - \xi_*}. \quad (56)$$

The main result on distributed event-triggered coordination is stated as follows.

Theorem 4 Suppose $\delta(t) = Ae^{-\theta t}$ with $A > 0$ and $0 < \theta < \theta_0$. Let L_0 satisfy

$$L_0 e^{2\theta L_0} (N-1) \left[\frac{(N-1)(4d_0+1)A_0^2}{\theta_0 - \theta} + 1 \right] < \frac{1}{2}. \quad (57)$$

Then System (1) with control law (53) achieves a GAC with $\tau_0 > 0$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proof. Define a function

$$M(t) \doteq \inf \{ \tau_k^i | t_k^i < t, i = 1, \dots, N; k = 0, 1, \dots \}$$

as the lower bound for the inter-event times before time t . $M(t)$ is obviously non-increasing. Moreover, for any $j \in \hat{\mathcal{N}}_i(k)$, j can be triggered at most $2L_0/M(t)$ times during $t \in [t_{k-1}^i, t_{k+1}^i)$ because $|t_{k+1}^i - t_{k-1}^i| \leq 2L_0$. Thus, based on the definition of $\mathcal{T}_i^j(k)$, we see that

$$\left| x_j(t_{\mathcal{T}_i^j(k)}^j) - x_j(t_{\mathcal{T}_j(t)}^j) \right| \leq \frac{2L_0}{M(t)} \delta(t_{k-1}^i) \leq \frac{2L_0}{M(t)} \delta(t) e^{2\theta L_0}, \quad t \in [t_{k-1}^i, t_{k+1}^i).$$

Then (54) implies

$$|\hat{w}_i(t)| \leq (N-1) \left[2 + \frac{2L_0 e^{2\theta L_0}}{M(t)} \right] \delta(t), \quad i = 1, \dots, N. \quad (58)$$

Noting that, for any $t \geq 0$, each arc in $\mathcal{G}_{\sigma(t)}$ will still be kept in the communication graph defined by neighbor sets $\hat{\mathcal{N}}_i(k)$ $i = 1, \dots, N; k = 1, \dots$. Hence the dwell time assumption still stands, and therefore, we can conclude from (36) that

$$\begin{aligned}\mathcal{H}(x(t)) &\leq (1 - \xi_*)^{\lfloor \frac{t}{K_0} \rfloor} \mathcal{H}(x^0) + (4d_0 + 1) \int_0^t (1 - \xi_*)^{\lfloor \frac{t}{K_0} \rfloor - \lceil \frac{\tau}{K_0} \rceil} (N - 1) \left[2 + \frac{2L_0 e^{2\theta L_0}}{M(\tau)} \right] \delta(\tau) d\tau \\ &\leq A_0 e^{-\theta_0 t} \mathcal{H}(x^0) + 2(N - 1)(4d_0 + 1) \left[1 + \frac{e^{2\theta L_0} L_0}{M(t)} \right] A_0^2 e^{-\theta_0 t} \int_0^t e^{\theta_0 \tau} \delta(\tau) d\tau \\ &= A_0 e^{-\theta_0 t} \mathcal{H}(x^0) + 2(N - 1)(4d_0 + 1) \left[1 + \frac{e^{2\theta L_0} L_0}{M(t)} \right] \frac{A_0^2 A}{\theta_0 - \theta} (e^{-\theta t} - e^{-\theta_0 t})\end{aligned}\quad (59)$$

Claim. $\tau_0 > 0$.

We prove the claim by contradiction. Assume that $\tau_0 = 0$. Then we have $\lim_{t \rightarrow \infty} M(t) = 0$. Without loss of generality, let us assume that $\tau_{k+1}^i = \delta(t_{k+1}^i) / |u_i(t_k^i)|$ for all $i = 1, \dots, N$ and $k = 0, 1, \dots$. From (55), we have

$$|u_i(t_k^i)| \leq (N - 1) \mathcal{H}(x(t_k^i)) + |\hat{w}_i(t_k^i)| \leq (N - 1) \mathcal{H}(x(t_k^i)) + (N - 1) \left[2 + \frac{2L_0 e^{2\theta L_0}}{M(t_k^i)} \right] \delta(t_k^i).$$

Now we can further conclude from (59) that

$$\begin{aligned}\tau_{k+1}^i &\geq \frac{\delta(t_k^i)}{(N - 1) \mathcal{H}(x(t_k^i)) + \left[2 + \frac{2L_0 e^{2\theta L_0}}{M(t_k^i)} \right] \delta(t_k^i)} \cdot e^{-\theta \tau_{k+1}^i} \\ &\geq \frac{AM(t_k^i)}{\left[A_0 \mathcal{H}(x^0) + \frac{2(N-1)(4d_0+1)A_0^2 A}{\theta_0 - \theta} + 2A \right] M(t_k^i) + 2A \left[\frac{(N-1)(4d_0+1)A_0^2}{\theta_0 - \theta} + 1 \right] L_0 e^{2\theta L_0}} \\ &\quad \times \frac{e^{-\theta \tau_{k+1}^i}}{N - 1}.\end{aligned}$$

Now we know that for any fixed number $0 < \mu < 1$, there exists $N_1 > 0$ such that when $k > N_1$,

$$\tau_{k+1}^i \geq \mu \cdot \frac{M(t_k^i)}{2L_0 e^{2\theta L_0} (N - 1) \left[\frac{(N-1)(4d_0+1)A_0^2}{\theta_0 - \theta} + 1 \right]} \cdot e^{-\theta \tau_{k+1}^i}. \quad (60)$$

Since $\tau_0 = 0$, there has to be $\tau_{k_0+1}^{i_0} \rightarrow 0$ as k_0 tends to infinity such that $\tau_{k_0+1}^{i_0} = M(t_{k_0}^{i_0} + \tau_{k_0+1}^{i_0})$. On the other hand, with (57), we can choose μ and k_0 sufficiently large to enforce

$$\mu \cdot \frac{1}{2L_0 e^{2\theta L_0} (N - 1) \left[\frac{(N-1)(4d_0+1)A_0^2}{\theta_0 - \theta} + 1 \right]} \cdot e^{-\theta \tau_{k_0+1}^{i_0}} > 1.$$

As a result, (60) will lead to

$$M(t_{k_0}^{i_0} + \tau_{k_0+1}^{i_0}) > M(t_{k_0}^{i_0}), \quad (61)$$

which contradicts the fact that $M(t)$ is non-increasing. The claim is proved.

With $\tau_0 > 0$, we further obtain

$$|\hat{w}_i(t)| \leq 2(N-1) \left[1 + \frac{L_0 e^{2\theta L_0}}{\tau_0} \right] \delta(t), \quad (62)$$

which guarantees GAC for system (1) immediately according to Proposition 1. The desired conclusion follows. \square

7 Conclusions

This paper focused on the robustness of continuous-time consensus algorithms of single integrator. We provided a precise answer to how much connectivity is required for the network to agree asymptotically based on noisy communications.

The idea of input-to-state stability and integral input-to-state stability inspired us to our definitions of robust consensus and integral robust consensus. We showed that uniform joint connectivity is critical with respect to robust consensus for general directed graphs; infinite joint connectivity is critical with respect to integral robust consensus. Upper bounds for the ϵ -convergence time were obtained as a straightforward result of the robustness analysis.

The results may have many applications since the assumptions we use are quite general. By a generalized integral version of the weight assumption, the dynamics can cover many models in both theoretical and application study. As an illustration, we studied distributed event-triggered coordination using the robust consensus inequality.

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